International Journal of Theoretical Physics, Vol. 30, No. 11, 1991

Stochastic Processes in Conformal Riemann–Cartan–Weyl Gravitation

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Received September 25, 1990

We construct the Riemann–Cartan geometries with torsion generated by the action of the conformal Weyl group. We study the wave operators associated to these structures, which, in addition to the usual Laplace–Beltrami operator, have a term which is a gradient vector field conjugate to the one-form given by the torsion potential derived from the Weyl group, and which we associate with a relativistic extension of the drift vector field in Nelson's construction of stochastic mechanics. In fact, our construction is valid for configuration spaces of any dimension. We sketch the construction of the stochastic processes on space-time generated by these operators, where the invariant measure is found to be defined by the conformal structure. We discuss briefly the relation with the theory of Dirichlet forms and D. Bohm's quantum potential in the theory of hidden variables, which in this setting acquire a gauge-geometric status previously unknown.

INTRODUCTION

Cartan's legacy in differential geometry comprises more than the construction of classical mechanics through his theory of integral invariants; the general theory of linear connections was formalized by Ehresmann (1950), from which the gauge theories later arose.

It is a theory which comprises a torsion tensor.

On encountering Einstein, Cartan emphasized that his theory was more general than the purely metric theory of Levi-Civita, and proposed an extension of Einstein's relativity (Cartan and Einstein, 1979), which is known today as the Einstein–Cartan theory (Hehl, 1980). Kibble and Sciama (1962) contributed to it in the 1960s, developing a gauge theory of the Lorentz group with torsion. Hehl *et al.* (1976) later associated torsion to a spin density, and de Sabbata and Gasperini made a pioneering project of relating

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0020-7748/91/1100-1497$06.50/0

torsion to the "internal" symmetries of particle physics, through a nonstatic propagating torsion (Hehl, 1980). Independently, the author together with Sternberg proved the central role this propagating torsion possesses in a gauge approach to the symplectic geometric description of the elementary classical relativistic systems with spin, classified by Souriau (1970; also see Sternberg and Ungar, 1978; Guillemin and Sternberg, 1984). It was found that a relativistic *spinless* particle moves as if the background field were torsionless.

Unrelated to this, Nelson has derived nonrelativistic quantum mechanics from the theory of Markov processes, and the subsequent theory, stochastic mechanics, is conceptually richer than the usual approach to quantum phenomena, and has been proposed by Glimm and Jaffe (1987) as an alternative construction to the program of constructive field theory.

Yet, in Nelson's theory as well as in the fundamental studies of the theory of Markov processes in general manifolds, there is lacking a fundamental geometrical principle from which these theories should naturally arise, even though the necessity of the general theory of connections due to Cartan plays a fundamental role in the mathematical theory (Ikeda and Watanabe, 1981; Elworthy, 1982; Azema and Yor, 1982).

In this paper we shall provide this unifying geometrical principle, as a theory of the gauge conformal Riemann–Cartan structures, with their associated wave operators the infinitesimal generators of the Markov processes in space-time. This point of view is original, to our knowledge, and, as we shall see, it allows for a most remarkable connection between the gravitational field described by a Poincaré gauge theory, stochastic mechanics, and quantum mechanics as described by the theory of Dirichlet forms.

To be precise, we shall construct a one-to-one correspondence between the conformal-Lorentz (or conformal-orthogonal) gauge theory, the diffusion processes associated to their wave operators, and the Dirichlet forms associated to them.

Therefore, this principle might provide for a fundamental geometrical and gauge perspective which has been lacking in quantum field theory.

In this paper we shall give a brief presentation of this principle and its above-mentioned relations.

1. THE RIEMANN–CARTAN–WEYL GEOMETRY OF POINCARÉ GAUGE THEORY

We shall assume from now on, unless otherwise stated, that all geometrical structures are infinitely differentiable, and that space-time M has dimension equal to 4.

The Riemann–Cartan geometry on space-time appears from a reduction of the bundle of Poincaré frames over space-time to that of the Lorentz frames. The fundamental geometrical object is the Cartan soldering form θ, a Lorentz [or $O(4)$] tensorial R^4-valued one-form on M, which allows for smooth identification of T_xM, the tangent space at $x \in M$, and the homogeneous space R^4 given by the quotient P/L, where P and L are the Poincaré and Lorentz [or $O(4)$], respectively. This soldering form together with the connection below describe the gravitational field (Cartan and Einstein, 1923; Rapoport-Campodónico and Sternberg, 1984*a*,*b*; Heyl, 1976). Thus, θ gives a (co)tetrad field $\theta^a_\alpha \, dx^\alpha$, with *inverse* $e^\alpha_a \, \partial/\partial x^\alpha$ in a coordinate system (x^α) of M, where $\alpha = 1, \ldots, 4$ and $a = 0, 1, \ldots, 3$ represent the indices of an anholonomic basis in R^4; thus,

$$\theta^a_\alpha e^\alpha_b = \delta^a_b \qquad \text{and} \qquad \theta^a_\alpha e^\beta_a = \delta^\beta_\alpha$$

If (g_{ab}) denotes a metric on R^4, we can define a metric g on M, by

$$g_{\alpha\beta} = g_{ab} \theta^a_\alpha \theta^b_\beta \tag{1.1}$$

which then has the same signature as (g_{ab}).

If we have a Lorentz (or orthogonal) linear connection on R^4, $\Gamma = (\Gamma^{ab}_\mu)$, then Γ is skew-symmetric in a, b; we assume Γ to be metric-compatible. From Γ we can define the space-time linear connection (Hehl *et al.*, 1976)

$$\Gamma^\alpha_{\beta\mu} = e^\alpha_a \theta^b_\beta \Gamma^a_{b\mu} + e^\alpha_a \partial_\mu \theta^a_\beta \tag{1.2}$$

which then is also metric-compatible, i.e., if ∇ denotes the exterior covariant differential with respect to the linear connection defined by (1.2), then

$$\nabla_\alpha g_{\beta\mu} = 0 \tag{1.3}$$

Thus, lengths of vector fields are preserved under parallel transport. This means that θ has reduced the bundle of linear frames to the orthogonal bundle (this is of great importance in assuring the *strong* Markov property for the stochastic processes we shall construct below; yet we shall not make this property explicit here).

What is essential to the connection on M defined by (1.3) is its nonsymmetric character, i.e., it has a nonzero torsion tensor

$$T^\alpha_{\mu\nu} = \tfrac{1}{2}(\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}) \tag{1.4}$$

This geometry is called the Riemann–Cartan (RC) structure.

Let us introduce a conformal structure on the tangent space of M.

We define the Weyl transformation on the soldering form by

$$W(\theta^a_\alpha) = \psi \theta^a_\alpha \tag{1.5a}$$

so that $W(e_a^\alpha)=(1/\psi)e_a^\alpha$, and a Weyl transformation on Γ (which by abuse of notation we denote by W as well as for the other derived transformations)

$$W(\Gamma_{b\mu}^a)=\Gamma_{b\mu}^a \tag{1.5b}$$

then we can *derive* the following transformation on the metric on M:

$$W(g_{\alpha\beta})=\psi^2 g_{\alpha\beta} \qquad \text{and} \qquad W(g^{\alpha\beta})=\psi^{-2}g^{\alpha\beta} \tag{1.6}$$

These are the well-known conformal transformations of the *metric* on M (Fulton *et al.*, 1962). In the above definitions, ψ is a function defined on M with values on $\mathbb{R}^+$, which initially we shall take to be smooth on any open neighborhood not containing the "node set of ψ" defined by

$$\{x\in M/\psi(x)=0\}$$

which is closed.

The Riemann–Cartan structure under the above transformations becomes

$$W(\Gamma_{\beta\mu}^\alpha)=\Gamma_{\beta\mu}^\alpha+\delta_\beta^\alpha\partial_\mu \ln\psi \tag{1.7}$$

with torsion tensor

$$T_{\beta\mu}^\alpha+\tfrac{1}{2}(\delta_\beta^\alpha\partial_\mu \ln\psi-\delta_\mu^\alpha\partial_\beta \ln\psi) \tag{1.8}$$

This shows that only the trace of the torsion tensor is conformally transformed, i.e., the 1-form $Q=Q_\mu\, dx^\mu=T_{\alpha\mu}^\alpha\, dx^\mu$ of the original connection is transformed as $W(Q)=Q+3/2d\ln\psi$.

The fact that is to be remarked is that one could in principle start with a *torsionless* and *flat* connection, say, such a Γ, with $(\theta_\alpha^a)=(\delta_\alpha^a)$ and $g=\mathrm{diag}(\pm 1, 1, 1, 1)$; and through a choice of conformal tetrads (1.5a), we henceforth introduce an RC structure. It is important to notice that this also introduces a *metric-compatible* connection. It is given by (we normalize the $\frac{3}{2}$ factor)

$$\Gamma_{\beta\mu}^\alpha=\left\{{\alpha \atop \beta\mu}\right\}+\tfrac{2}{3}(\delta_\beta^\alpha\partial_\mu \ln\psi-g_{\beta\mu}g^{\gamma\alpha}\partial_\gamma \ln\psi) \tag{1.9}$$

where $\left\{{\alpha \atop \beta\mu}\right\}$ are the coefficients of the Levi-Civita connection associated to the metric defined by (1.1). Then, $Q=d\ln\psi$, the logarithmic differential of the scale field ψ, is a Weyl one-form of an RC metric-compatible structure.

This distinguishes these RC structures, produced by the general action of the conformal group, from the usual Weyl geometry produced by the transformations on the space-time metric (1.6). In the latter, it is the Weyl one-form which precisely expresses the lack of preservation of lengths under parallel Weyl transport (Fulton *et al.*, 1962). So the introduction of these structures solves a long pending problem of compatibility of the RC structures with the local action of the Weyl group (see, e.g., Katanaev and Volovich, 1990).

Therefore, this geometry, which we shall call Riemann–Cartan–Weyl (RCW), has *no historicity problem*, which moved Einstein to reject Weyl's attempt to construct the first gauge theory in which he associated the Weyl form to the electromagnetic field, this in spite of Q not being a complex field (yet, a nonobvious association).

We remark that our above constructions can be carried out for the case of a general configuration space M of dimension n, on taking instead of the Poincaré group, the group given by the semidirect sum $O(n)+\mathbb{R}^n$.

As we already claimed, we intend to construct an extension of stochastic mechanics (Nelson, 1985). For this we need to characterize the infinitesimal generators of the stochastic processes, which we later relate to the Dirichlet operators of quantum mechanics (Blanchard *et al.*, 1987).

We shall study, then, the wave operator associated to the RC structures.

Henceforth, in this section the dimension of M will be arbitrary n. Let ω be an arbitrary p-form on M. Locally

$$\omega = 1/p!\, \omega_{\alpha_1\cdots\alpha_p}\, dx^{\alpha_1}\wedge\cdots\wedge\ \wedge\, dx^{\alpha_p}$$

Then,

$$\nabla\omega = 1/(p+1)!\left(\nabla_\alpha\omega_{\alpha_a\cdots\alpha_p} - \sum_{k=1}^{p}\nabla_{\alpha_k}\omega_{\alpha_1\cdots\alpha_{k-1}\alpha\alpha_{k+1}\cdots\alpha_p}\right) dx^\alpha\wedge dx^{\alpha_1}\wedge\cdots dx^{\alpha_p}$$

is a $(p+1)$-form, which can be uniquely decomposed as $\nabla_g\omega + d_c\omega$, where ∇_g denotes the covariant derivative with respect to the Levi-Civita connection associated to g, so that

$$\nabla_g\omega = 1/(p+1)!\,(\partial_\alpha\omega_{\alpha_1\alpha_2\cdots\alpha_p} - \partial_{\alpha_1}\omega_{\alpha\alpha_2\cdots\alpha_p} - \partial_{\alpha_2}\omega_{\alpha_1\alpha\cdots\alpha_p} - \cdots - \partial_{\alpha_p}\omega_{\alpha_1\cdots\alpha_{p-1}\alpha})\, dx^\alpha\wedge dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_p}$$

and

$$d_c\omega = 2/(p+1)!\left(\sum_{m=1}^{p}\omega_{\alpha_1\cdots\alpha_{m-1}\beta\alpha_{m+1}\cdots\alpha_p}T^{\beta}_{\gamma\alpha_m} + \sum_{l<k}^{p}\omega_{\alpha_1\cdots\alpha_{l-1}\beta\alpha_{l+1}\cdots\alpha_{k-1}\gamma\alpha_{k+1}\cdots\alpha_p}T^{\beta}_{\alpha_l\alpha_k}\right) dx^\gamma\wedge dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_p}$$

The covariant codifferential $\delta\omega$ of ω is a $(p-1)$-form given by

$$\delta\omega = -1/(p+1)!\, g^{\beta\alpha}\nabla_\beta\omega_{\alpha\alpha_2\cdots\alpha_p}\, dx^{\alpha_2}\wedge\cdots\wedge dx^{\alpha_p}$$

The covariant codifferential of a function is defined to be zero.

We define the wave operator $\mathscr{L}$ of the RC structure as

$$\mathscr{L} = \delta\nabla + \nabla\delta \tag{1.10}$$

Hence, if ϕ denotes a function on M, $\mathscr{L}(\phi) = \delta\nabla\phi$, which can still be written as $\mathscr{L}(\phi) = \delta\, d\phi$.

δ can be defined from *, the (extension) of the duality Hodge operator

$$\delta\omega = (-1)^{p^{*-1}}\, \nabla{*}\omega \tag{1.11}$$

$[*^{-1} = (-1)^{p(n-p)}*]$, where

$$*\omega = 1/(n-p)!\; e_{\alpha_1\alpha_2\cdots\alpha_{n-p}\beta_1\cdots\beta_p} Q^{\beta_1\cdots\beta_p} \tag{1.12}$$

with $e_{\alpha_1\cdots\alpha_n}$ being the covariant components of the unit tensor field and

$$Q^{\beta_1\cdots\beta_p} = g^{\beta_1\gamma_1} g^{\beta_2\gamma_2} \cdots g^{\beta_p\gamma_p} \omega_{\gamma_1\cdots\gamma_p} \tag{1.13}$$

are the components of the conjugate tensor field Q of ω. For example, if $\pi = \pi_\alpha\, dx^\alpha$ is an arbitrary one-form, its conjugate vector field is $\pi^\beta \partial_\beta$, with $\pi^\beta = g^{\beta\gamma}\pi_\gamma$.

Let us take now the $(n-1)$-form dual to the 1-form π,

$$\omega = {*\pi} = 1/(n-1)!\; \omega_{\alpha_1\alpha_2\cdots\alpha_{n-1}}\, dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{n-1}}$$

where

$$\omega_{\alpha_1\alpha_2\cdots\alpha_{n-1}} = e_{\alpha_1\alpha_2\cdots\alpha_{n-1}\alpha_n} \pi^n$$

Applying the covariant differential operator ∇, we obtain

$$\nabla_g \omega = (-1)^{n-1} \partial_\alpha (|\det g|^{1/2} \pi^\alpha) \wedge dx^1 \wedge \cdots \wedge dx^n$$

and

$$\begin{aligned} d_c\omega &= (-1)^{n-1} 2|\det g|^{1/2} \pi^\alpha T^\beta_{\alpha\beta}\, dx^1 \wedge \cdots \wedge dx^n \\ &= (-1)^{n-1} 2|\det g|^{1/2} g(\hat{\pi}, \hat{Q})\, dx^1 \wedge \cdots \wedge dx^n \end{aligned}$$

where $\hat{\pi}$ and $\hat{Q}$ denote the vector fields conjugate to the 1-forms π and $Q = T^\alpha_{\alpha\beta}\, dx^\beta$, respectively. Therefore,

$$\nabla\omega = -(\partial_\alpha(|\det g|)^{1/2}\pi^\alpha) + 2|\det g|^{1/2} g(\hat{\pi}, \hat{Q}))\, dx^1 \wedge \cdots \wedge dx^n,$$

Thus, the generalized divergence $\mathrm{Div}(\pi)$ of π is a scalar form given by

$$\mathrm{Div}\ \pi = |\det g|^{-1/2}\partial_\alpha(|\det g|^{1/2}\pi^\alpha) + 2g(\hat{\pi}, \hat{Q})$$

Finally, the generalized Laplacian of a function ϕ is given by the invariant expression

$$\mathscr{L}(\phi) = -\mathrm{Div}\ d\phi = -(|\det g|^{-1/2}\ \partial_\alpha(|\det g|^{1/2}\partial^\alpha \phi) + 2g(d\hat{\phi}, \hat{Q})) \quad (1.14)$$

We note that only the torsion-trace one-form is involved in the second term, while the first one is nothing else than the usual Laplace–Beltrami operator Δ_g associated to the space-time metric applied to ϕ. One can write $\mathscr{L}(\phi)$ in the simpler form

$$\mathscr{L}(\phi) = -(\Delta_g + 2Q^\gamma\partial_\gamma)(\phi) \quad (1.15)$$

The torsion-trace 1-form need not be exact. Yet, on assuming this exactness, say $Q = d\ln\psi$, for some nonnegative function ψ on M, then (1.15) is the expression for the wave operator of a Riemann–Cartan–Weyl defined by the choice of a conformal element ψ. This is

$$\mathscr{L}_\psi(\phi) = -[\Delta_g + 2\partial^\gamma(\ln\psi)\partial_\gamma](\phi) \quad (1.16)$$

where we have written $\mathscr{L}$ as $\mathscr{L}_\psi$ to stress its dependence in the *conformal element* ψ defining the RCW structure.

Therefore, the wave operator for the RCW structures has, in additional to the usual propagation term, a coupling term which is a first-order operator given by the conjugate of twice the Cartan–Weyl form $d\ln\psi^2$.

Yet, in quantum physics, to define correctly the ground state of a system under consideration, one is interested in Hamiltonian operators given by one-half the Laplacian (the heat operator), so we shall focus our attention on the operator

$$H_\psi = -\tfrac{1}{2}\mathscr{L}_\psi = \tfrac{1}{2}\Delta_g + \partial^\alpha(\ln\psi)\partial_\alpha \quad (1.17)$$

This natural renormalization will turn to be of fundamental importance in the sequel, for the construction of a relativistic extension of stochastic mechanics.

2. THE STOCHASTIC PROCESSES ASSOCIATED TO THE RIEMANN–CARTAN–WEYL GEOMETRIES

In Nelson's (1967, 1985) classical work on the construction of quantum mechanics from the theory of stochastic processes, specifically, the Markov processes, wave mechanics is constructed in terms of measures in the space of paths on configuration space, in principle R^3 or a three-dimensional space manifold provided with a Levi-Civita connection. Still, it considers the configuration space for a system of nonrelativistic particles. So, in principle, it is a nonrelativistic theory, in which time is associated to the evolution parameter of the quantum system. The system itself is described by a Markov diffusion process $\xi=\xi(t)$, $t\in\mathbb{R}$, which is constructed from the assumptions of an initial probability density for ξ, and the infinitesimal generator of the process, a second-order differential operator described by a Hamiltonian of the form $H=\frac{1}{2}\Delta_g+b$, where b is a first-order differential operator with coefficients which depend on both the space and time coordinates. In the theory of stochastic processes, the Laplace–Beltrami operator is the generator of the inherently stochastic character of the process (if $b\equiv 0$, it is a process described by the well-known Wiener measure) and b is the drift or velocity vector field, thought of as a classical contribution to the dynamics of the system, perturbed by the stochastic term. In Nelson's theory, b is *unrelated to any geometrical structure*, and it is assumed separately for generation of the process. More precisely, one has to solve for the drift in solving for a scalar field satisfying Schrödinger's equation, thus turning a nonlinear theory of diffusion (the Fokker–Planck equations) into a linear one. Thus, it is fundamental for his construction of the wave functions, essentially the above conformal factors, that the drift vector field be assumed gradient-like (Blanchard *et al.*, 1987).

All further developments of stochastic mechanics (SM)—Aldrovandi *et al.*'s (1990) and Albeverio and Høegh-Krohn's (1979) program of constructive quantum field theory—carry this imprint.

At this stage of our presentation, it seems natural to pause to reflect on the fact that in our construction of the RCW structures we have a conceptually richer structure than the usual one contained in the known construction of SM.

Indeed, if we take for infinitesimal generators of the stochastic processes the Hamiltonian operators H_ψ constructed from the wave operators of the RCW structures, we have: first, Minkowski or Euclidean time is incorporated into the configuration space, and *a priori* should not be confused with the evolution parameter of the stochastic processes {which we shall denote from now on as τ, with $\tau\in[0,\infty)$; this proper-time parameter should be thought of as a Kaluza–Klein $(n+1)$-coordinate}; second, we are in a position to describe a relativistic system of particles, as our construction allows us to deal with arbitrary dimensions; finally, if we look for a (we remark) τ-stationary measure on the paths of the diffusion, we shall see that the statistical description of the theory becomes purely geometrical.

To start with, *by construction* of the RCW structures, the drift is of gradient type.

Certainly, b coincides with the conjugate vector field to $d\ln\psi$. There is no accident to this remarkably perfect matching.

The key idea of construction of SM is that of parallel transport of a diffusion process, so the question of the linear connection to describe this parallel transport *is* the central issue. The choice of a Levi-Civita connection not only is of negative character (a torsionless connection), but also *chooses* a preferred system of coordinates, the so-called *normal coordinates*, for which a well-known theorem of differential geometry assures that a general linear connection reduces to a Levi-Civita one. In other words, this special type of parallel transport conflicts with the principle of relativity. So, whatever one has to define as a parallel transport, be this along an a.s. continuous but nowhere differentiable path as in Brownian motion, or along a smooth path as in differential geometry, it appears that the RCW structures will *precisely* be the natural ones defining the required *law* of parallel transport. What one needs is to take into account the locally infinite variation of the paths of Brownian motion, i.e., a specific set of rules of calculation which will do the job of the usual differential calculus on manifolds. This is the so-called *Ito stochastic calculus* (Ikeda and Watanabe, 1981; Williams, 1981), which we shall omit due to limitations of space.

The stochastic processes we shall build constitute a class characterized by *both* the Markov property and the continuity of its paths in M; they are usually called *diffusion processes*. We shall give a formal (but incomplete) description of them. A more complete formal treatment of some of its aspects can be extracted from Meyer and Zheng (1985) and a forthcoming paper by the author.

Let M be topological space (one usually takes the one-point compactification on aggregating a terminal point at infinity on which the diffusion falls if it explodes, but for brevity, we shall not make explicit this possibility). Let $\mathscr{W}(M)$ be the set of all continuous functions $w\colon \mathbb{R} \to M$. A *Borel cylinder set* in $\mathscr{W}(M)$ is defined for a sequence of positive real numbers $\tau_1 < \tau_2 < \cdots < \tau_n$ and a Borel subset A in $M^n = M \times \cdots \times M$ n times as $\pi^{-1}_{\tau_1,\ldots,\tau_n}(A)$, where $\pi_{\tau_1,\ldots,\tau_n}(w) = (w(\tau_1), \ldots, w(\tau_n))$. We recall that a Borel subset in M is any set in the smallest σ-field containing all open sets. Let $\mathscr{O}(\mathscr{W}(M))$ be the σ-field generated by all cylinder sets, and let $\mathscr{B}_\tau(\mathscr{W}(M))$ be the σ-field generated by all cylinder sets up to time τ. A family of probabilities $\{P_x, x \in M\}$ on $\{\mathscr{W}(M), \mathscr{B}(\mathscr{W}(M))$ is called a *Markovian system* if it satisfies the following conditions:

(i) $P_x\{w\colon w \in \mathscr{W}(M), w(0) = x\} = 1$, $\forall x \in M$.

(ii) $M \ni x \to P_x(A)$ is Borel measurable, for each $A \in \mathscr{B}(\mathscr{W}(M))$.

(iii) $\forall \tau \geq s$, $A \in \mathscr{B}_s(\mathscr{W}(M))$, and Γ a Borel subset in M,

$$P_x(A \cap \{w\colon w(\tau) \in \Gamma\}) = \int_A P_{w'(s)}\{w\colon w(\tau - s) \in \Gamma\} P_x(dw'), \quad \forall x \in M$$

We set $P(\tau, x, \Gamma) = P_x\{w\colon w(\tau) \in \Gamma\}$. The family $\{P(\tau, x, \Gamma)\}$ is called the *transition probability of a Markovian system*. By successive application of (iii) we get

$$P_x[w(\tau_1)\in A_1, \ldots, w(\tau_n)\in A_n]$$
$$= \int_{a_1} P(\tau_1, x, dx_1) \times \int_{A_n} P(\tau_2-\tau_1, x_1, dx_2)$$
$$\times \cdots \times \int_{A_n} P(\tau_n-\tau_{n-1}, x_{n-1}, dx_n)$$

for $0<\tau_1<\cdots<\tau_n$, and $A_1, \ldots, A_n \in \mathscr{B}(M)$, so we can see that two Markovian systems defined on M with the same transition probability coincide.

A stochastic process $\xi=(\xi(\tau))$ on M, i.e., a $\mathscr{W}(M)$-valued random variable [i.e., for fixed $\tau\geq 0$, the map $\mathscr{W}(M)\ni w \to \xi(\tau)(w)=\xi(\tau, w)$ is a random variable with values in M], where $(\mathscr{W}(M), \mathscr{B}(\mathscr{W}(M), P))$ is a probability space with a probability measure P, is called a *diffusion process*, if there exists a Markovian system $\{P_x, x\in M\}$ on $(\mathscr{W}(M), \mathscr{B}(\mathscr{W}(M)))$ such that, for almost all ω, the sample paths $[\tau \to \xi(\tau)]\in\mathscr{W}(M)$ and the probability law on $\mathscr{W}(M)$ (i.e., the image measure) of $[\tau \to \xi(\tau)]$ coincide with $P_\mu(\cdot)_M=\int P_x(\cdot)\,\mu(dx)$, where μ is the Borel measure on M defined by $\mu(dx)=P\{w\colon \xi(0, w)\in dx\}$.

We shall say that the operator H is the *infinitesimal generator* of a diffusion process $\xi=(\xi(\tau))$ (or that $\{P_x, x\in M\}$ is *determined* by H) if the *stochastic derivative* Df of any twice differentiable bounded function with bounded derivatives f on M satisfies the condition

$$Df(\xi(\tau))=\lim_{h\to 0^+} 1/h\, E_{\xi(\tau)}(f(\xi(\tau+h))-f(\xi(\gamma)))=(Hf)(\xi(\tau)) \quad (2.1)$$

where $E_{\xi(\tau)}$ denotes the expected value with respect to $P_{\xi(\tau)}$.

We are interested in M being space-time or a configuration space of arbitrary dimension n, and the diffusion processes defined by the infinitesimal generators given by $H=H_\psi=-\frac{1}{2}\mathscr{L}_\psi$, which we can locally write as

$$H=\tfrac{1}{2}a^{\alpha\beta}\partial^2_{\alpha\beta}+\partial^\alpha \ln\psi\,\partial_\alpha \quad (2.2)$$

Let $\sigma=(\sigma^\beta_\alpha)$ be a square root of Δ_g, i.e., $x \to \sigma(x)$ is continuous and the coefficients $a^{\alpha\beta}(x)$, $x\in M$, are given by $a^{\alpha\beta}=\sigma^\alpha_\gamma\sigma^\beta_\delta g^{\gamma\delta}$. This is a coordinate-dependent construction and nonunique; this will not affect the unicity of the diffusion (Friedlin, 1985).

If ψ is constant, so that the drift is zero, then the diffusion ξ is standard Brownian motion with $P_x=W_x$, the Wiener measure on M starting at x and initial distribution $\mu(dx)$.

Thus, in the general case we might expect a departure of the Gaussian measures one usually encounters in quantum field theory, produced by ψ.

We now consider the following (homogeneous in proper time τ) stochastic differential equation for $\xi = (\xi(\tau))$:

$$d\xi^{\alpha}(\tau) = \sigma^{\alpha}_{\beta}(\xi(\tau))\, dB^{\beta}(\tau) + b^{\alpha}(\xi(\tau))\, d\tau, \qquad \alpha = 1, \ldots, n \tag{2.3}$$

where $dB = (dB^{\alpha})$ denotes a Brownian motion on M, so that $E_{\xi(\tau)}(dB^{\alpha}(\tau)) = 0$, and the covariance satisfies $E_{\xi(\tau)}(dB^{\alpha}(\tau) \cdot dB^{\beta}(\tau)) = g^{\alpha\beta}\, d\tau$.

It is a theorem (Ikeda and Watanabe, 1981), in the case that Δ_g is an elliptic operator, so that g is a Riemannian metric (which we can think of as providing a Euclidean structure on TM), and σ and b are continuous and Lipschitz bounded on M, that for every $x \in M$ there exists a unique solution of (2.3) such that $\xi(0) = x$, and the probability law on $\mathscr{W}(M)$ of the diffusion ξ is determined by H. The probability law of $\xi(0)$ coincides with μ, i.e., $P\{\xi(0) \in A\} = \mu(A)$, where μ is the given probability measure on M. The condition on the metric can be relaxed to g being nonnegative definite (still not including the hyperbolic case) (Friedlin, 1985). Due to the zeros of ψ, b will be singular, yet the unicity of the diffusion can be constructed by assuming b to be locally bounded (Friedlin, 1985; Carlen, 1984; Durrett, 1984). We shall make this precise below.

Thus, we have, in principle, a one-to-one correspondence between the RCW structures and the diffusion processes generated by H_{ψ}.

Let $\{P_x : x \in M\}$ be the diffusion process determined by H. The *transition semigroup* T_{τ} of the diffusion generated by H is defined by

$$(T_{\tau} f)(x) = E_x(f(\xi(\tau))) = \int_M f(\xi(\tau))\, P_x(dw) \tag{2.4}$$

for f a bounded, continuous function with continuous bounded derivatives. The function $u = u(\tau, x) = E_x(f(\xi(\tau)))$ is the unique solution of the *heat equation*

$$\frac{\partial u}{\partial \tau} = Hu \tag{2.5}$$

with initial condition given by $\lim_{\tau \to 0, y \to x} u(\tau, y) = f(x)$. It can be proved that T_{τ} and H commute: $T_{\tau} H = H T_{\tau}$, so that we can write $T_{\tau} = e^{\tau H}$, which, due to the Markov property of the diffusion, yields a semigroup of operators: $T_{\tau + \tau'} = T_{\tau} T_{\tau'}$.

If $H = H_{\psi} = -\frac{1}{2}\mathscr{L}_{\psi}$, we obtain a family of *semigroups of "Schrödinger" operators* associated to the RCW structures defined by the choice of the conformal class defined by ψ.

In quantum physics, in the functional integral point of view, one is interested in the transition semigroups defined by infinitesimal generators of the form $\frac{1}{2}\Delta_g + V$, where V denotes a multiplication operator by a potential function, which one expresses through a Feynman–Kac formula.

In fact, the theory of diffusion processes assures that the potential perturbative factor can be assimilated to the diffusion process defined by $\frac{1}{2}\Delta_g$, as an exponential factor in the path-integral representation of the Schrödinger operator in terms of the Wiener measure (Glimm and Jaffe, 1987; Durrett, 1984).

There is a similar path-integral representation in the case of the general diffusion processes defined by the RCW structures. This is done through the so-called *Girsanov–Martin–Cameron* transformation; we shall present it below.

Returning to the problem of construction of the diffusion processes, we shall say that a Borel measure $\mu(dx)$ on M is an *invariant measure* by the diffusion process defined by H, if, for $\tau \geq 0$ and f bounded continuous on M,

$$\int (T_\tau f)(x)\, \mu(dx) = \int f(x)\, \mu(dx) \tag{2.6}$$

The diffusion $\{P_x, x \in M\}$ will be called *symmetrizable* if there exists a Borel measure $v(dx)$ on M such that for any f, g bounded continuous on M

$$\int (T_\tau f)(x) g(x)\, v(dx) = \int f(x)(T_\tau g)(x)\, v(dx) \tag{2.7}$$

It is easy to see that the measure for which the diffusion process is symmetrizable is invariant.

It can be proved that $\mu(dx)$ is invariant for the process generated by H if and only if

$$\int (Hf)(x)\, \mu(dx) = 0 \tag{2.8}$$

for every smooth function of compact support f on M.

We define an inner product on the smooth functions of compact support on M by

$$(f, g) = \int f(x) g(x)\, dx \tag{2.9}$$

where dx is the Riemannian volume element, so that $dx = |\det g|^{1/2}\, d^4x$. By integration by parts we get that $(Hf, h) = (f, H^*h)$, where H^* is the adjoint operator of H, so that

$$H^*h = \frac{1}{2}\Delta_g h - \operatorname{div}(h \cdot d \ln \psi) \tag{2.10}$$

We can restate the condition for invariance of the measure μ in terms of H^*. Indeed, (2.10) is equivalent to μ being a weak solution (in the sense of the theory of generalized functions) of the partial differential equation $H^*\mu = 0$.

Assuming that Δ_g is an elliptic operator, then H and in consequence H^* are elliptic, too, and in consequence any solution of the equation $H^* v = 0$ is of the form $\mu = \phi \cdot dx$, for ϕ a smooth function vanishing on the zeros of ψ. In fact, ϕ is precisely an eigenfunction corresponding to the largest eigenvalue $\lambda = 0$ of the eigenvalue problem $(H^* - \lambda)\phi = 0$; it is simple and its associated eigenspace is of the form $\{c\phi_0^*(x), c \in \mathbb{R}^+\}$, and in consequence all invariant measures for H are of the form $\mu = (c\phi_0^*)\, dx$, for some constant $c > 0$.

Let us determine μ precisely in the case of $H = H_\psi$. Choose a smooth function $U(x)$ on M such that $\mu = e^{-U}\, dx$ is an invariant measure for H, i.e., $H^*(e^{-U}) = 0$. Since $H^*(e^{-U}) = -\frac{1}{2}\delta_g d(e^{-U}) + \delta_g(e^{-U} d \ln \psi)$, where δ_g denotes the codifferential of ∇_g, therefore $-\frac{1}{2} de^{-U} + e^{-U} d \ln \psi = 0$, and $U = -\ln \psi^2$.

Therefore, we have proved that $\psi^2\, dx$ gives an invariant density for the diffusion processes generated by $(-\frac{1}{2})\mathscr{L}_\psi$. *The square of the conformal factor together with the Riemannian volume element determine a unique invariant density for the RCW diffusion processes, apart from the node set of* ψ $\{x \in M: \psi(x) = 0\}$. In fact, it can be proved that the diffusion is symmetrizable with respect to this invariant density.

Due to the origin of ψ as a local $\mathbb{R}^+$-symmetry, these singularities *do exist*. It is a theorem due to Nelson (1985) that the diffusion process *does not penetrate the node set, which can then be thought of as a barrier for the diffusion process*, i.e., the probability of penetration of the node set by the diffusion is nil. For a proof of this in terms of capacities in the theory of Dirichlet forms see Blanchard *et al.* (1987).

These barriers have a natural interpretation. They are associated to a very general phenomenon of *charge quantization, which can easily be described in terms of the principle of the argument in the theory of complex variables*. It is interesting also to remark that these nodes can be described in terms of Thom's theory of catastrophes. We shall give a description of these facts elsewhere. For an interesting list of examples of these barriers, see Blanchard *et al.* (1987).

Let us assume that ψ belongs to the Hilbert space $L^2(dx)$ of square-integrable functions on M with respect to dx. Let $\rho\, dx$ be the invariant probability density defined by $C\psi^2\,dx$, where $C^{-1}= \psi^2\,dx$. The semigroup $\exp(\tau H_\psi)$ can be defined on $L^2(\rho\,dx)$ by

$$(\exp(\tau \cdot H_\psi)f)(x) = E_{\xi(0)}(f(\xi(\tau))) = \int f(x)\rho(x)\,dx$$

for any $\tau \geq 0$, where $\xi(\tau)$ is a solution of (2.3). In fact, the infinitesimal generator of the semigroup is defined on all smooth, compact, supported functions which are zero on a neighborhood of the zeros of ψ, so that it is the Friedrichs self-adjoint extension of H_ψ, which we shall still denote by H_ψ; the quadratic form on $L^2(\rho\,dx)$ with respect to which this extension is defined is the one associated to the bilinear symmetric form given by (minus) the so-called Dirichlet form (Carmona, 1979)

$$(f, \tilde{f}) \to -\tfrac{1}{2}\int g(df, d\tilde{f})\rho\,dx \tag{2.11}$$

f and $\tilde{f}$ smooth functions on M, with compact support on the complement of N.

Let us make this more precise. Since $\psi \in L^2(dx)$, the operator multiplication by ψ is densely defined and self-adjoint, with domain $L^2(\rho\,dx)$. We define the conformal dependent unitary map

$$C_\psi\colon\quad L^2(\rho\,dx) \to \psi L^2(dx), \qquad \phi \to \psi\phi \tag{2.12}$$

with inverse

$$C_\psi^{-1}\colon\quad \psi L^2(dx) \to L^2(\rho\,dx), \qquad f \to \psi^{-1}f$$

Note that $\psi L^2(\rho\,dx) = \{\psi f; f \in L^2(\rho\,dx)\} \subseteq L^2(dx)$, and the contention is equality if and only if the node set of ψ has Lebesgue measure equal to zero, i.e., $\psi > 0$ dx-almost everywhere.

Let Ω be the open set given by the complement of the node set N. For f and $\tilde{f} \in C_0^1(\Omega)$ (where from now on the subscript 0 stands for compact support) consider the symmetric bilinear form

$$\varepsilon(f, \tilde{f}) = -\tfrac{1}{2}\int g(df, d\tilde{f})\rho\,dx \tag{2.13}$$

This is a local Markovian symmetric form (Fukushima, 1980). Let $\mathscr{H}_\rho(\Omega)$ be the closed subspace of $L^2(\rho\,dx)$ obtained by closing $C_0^2(\Omega)$ in the $L^2(\rho\,dx)$ norm.

Let us assume that—we shall give explicit conditions for this below—the restriction of ψ to Ω is such that the quadratic form $q(f) = \varepsilon(f, f)$ is closable in $\mathscr{H}_\rho(dx)$:

$$\text{if } q(f_n - f_m) \xrightarrow[n,m\to\infty]{} 0 \text{ and } (f_n, f_n)_{L^2(\rho dx)} \xrightarrow[n\to\infty]{} 0, \text{ then } q(f_n) \xrightarrow[n\to\infty]{} 0$$

for any $\{f_n, n\in\mathbb{N}\}$ in $\mathscr{H}_\rho(\Omega)$. Then, between all the closed extensions of ε (the Dirichlet forms), we can take the smallest closed extension (the Friedrichs form) $\bar{\varepsilon}$, with associated quadratic form $\bar{q}$, defined by

$$\bar{\varepsilon}(f, \tilde{f}) = -\tfrac{1}{2}\int g(\bar{d}f, \bar{d}\tilde{f})\rho\, dx \tag{2.14}$$

where now $\bar{d}$ denotes the operator closure in $\mathscr{H}_\rho(\Omega)$ of the restriction of the exterior differential to $C_0^2(dx)$, where the exterior differential is thought of as an operator taking functions in $L^2(\rho\, dx)$ into 1-forms with coefficients in $L^2(\rho\, dx)$.

It is a fundamental theorem of the theory of Dirichlet forms (Fukushima, 1980) that there is one-to-one correspondence between the family of closed symmetric forms on $L^2(\rho\, dx)$ and the family of nonnegative-definite self-adjoint operators H_ρ on $L^2(\rho\, dx)$. We are particularly interested in the Friedrichs self-adjoint extension defined by the choice of domain given by $\mathscr{H}_\rho(\Omega)$, so that H_ρ is determined uniquely by

$$\bar{\varepsilon}(f, \tilde{f}) = (H_\rho f, \tilde{f})_{\mathscr{H}_\rho(\Omega)} \tag{2.15}$$

with domains of $\bar{\varepsilon}$ and of H_ρ equal to $\mathscr{H}_\rho(\Omega)$.

If ψ is such that $\hat{d}\ln\psi\in L^2_{\rm loc}(\Omega)$, then on $C_0^2(\Omega)$ we find that H_ρ equals H_ψ given by (1.17), i.e., there is a one-to-one correspondence between the smallest closed extensions $\bar{\varepsilon}$ defined by (2.14) and the Friedrichs extensions of the RCW wave operators (so, in the following, we shall denote them indistinguishably by H_ψ).

Let us explore further the relation between the Dirichlet forms and the RCW wave operators. This will yield a most remarkable one-to-one correspondence between the RCW structures and quantum mechanics as a theory of quadratic forms (Simon, 1971).

Due to the conformal unitary equivalence between $L^2(\rho\, dx)$ and $\psi L^2(\rho\, dx)$, to the closed subspace $\mathscr{H}_\rho(\Omega)$ of $L^2(\rho\, dx)$ there corresponds a closed subspace of $\psi\mathscr{H}_\rho(\Omega)\subseteq L^2(dx)$, and to H_ψ there corresponds a Hamiltonian operator D_ψ defined on $\psi\mathscr{H}_\rho(\Omega)$ by $D_\psi := C_\psi H_\psi C_\psi^{-1}$.

If we assume that $\Delta_g\psi\in L^2_{\rm loc}(\Omega)$, then the operator $V_\psi = \Delta_g\psi/\psi$ is densely defined in $L^2(\Omega, dx)$. For f in $C_0^2(\Omega)$ we have

$$D_\psi(\psi f) = C_\psi(H_\psi f) = \tfrac{1}{2}(\Delta_g - V_\psi)\psi f \tag{2.16}$$

i.e., on $\psi C_0^2(\Omega)$ we have $D_\psi = \frac{1}{2}(\Delta_g - V_\psi)$; we recognize V_ψ as the well-known quantum potential in Bohm's (1952) theory of hidden variables, which is essential also in stochastic mechanics, as we shall see below (Nelson, 1985; Blanchard *et al.*, 1987).

It is clear that ψ, as a generalized function, is the ground state of the operator D_ψ, which we have called Hamiltonian, as it possesses the usual decomposition into kinetic plus potential terms. We remark that in the context of the above formulation, the quantum potential V_ψ appears as a conformal transformation inside of the Dirichlet form of both the torsion-potential drift and of the *ground-state* invariant density of $\bar{\varepsilon}$ defined by $\psi^2\,dx$ to the Riemannian density.

If, additionally, the generalized 1-forms $d\psi$, $d\ln\psi$, and the generalized $\Delta_g\psi/\psi$ function all belong to $L^2_{\text{loc}}(\Omega)$, then D_ψ is still defined as in (2.16) on $C_0^2(\Omega)\cup(\psi C_0^2(\Omega))$ as well as H_ψ (Albeverio *et al.*, 1977).

We can summarize the above considerations in the following theorem: There are one-to-one correspondences between the RCW structures, their Friedrichs self-adjoint wave operators, the diffusion processes determined by them, the Friedrichs self-adjoint Dirichlet forms, and the Hamiltonian operators defined as sums of quadratic forms. This is regardless of dimension.

This is a most remarkable connection between the structures underlying the conformal gauge theory of gravitation and quantum mechanics.

To complete our construction of the diffusion processes, we must give the transition density $p_\tau^\psi(x, y)$ with respect to the Lebesgue measure, so that $\exp(\tau H_\psi)$ can be represented for any f in $\mathscr{H}_\rho(\Omega)$, in terms of the *Riemannian density* determined by g,

$$(\exp(\tau H_\psi)f)(x)=\int p_\tau^\psi(x,y)f(y)|\det g(y)|^{1/2}\,dy^1\wedge\cdots\wedge dy^4 \quad (2.17)$$

Due to the unitary equivalence of H_ψ and D_ψ, what we shall do is *determine* p_τ^ψ in terms of a Wiener process generated by the Hamiltonian D_ψ which, we stress, possesses the usual form of a Laplacian plus perturbative term so we can apply to it the usual Feynman–Kac formula, but, as we shall see, on the coordinate functions of the general process ξ.

Let us do this for the case in which g is the Euclidean metric, so that Δ_g is the wave operator on $\mathbb{R}^n$, so that in (2.13), σ is the identity matrix and B_τ is the standard Wiener process W_τ with transition density given by (2.24).

Applying the Feynman–Kac formula (Friedlin, 1985; Durrett, 1984; Rapoport-Campodónico and Tilli, 1987) we obtain

$$(\exp(\tau H_\psi)f)(x)=\psi(x)^{-1}E_{W_x}\left(\exp\left[-\int_0^\tau V_\psi(\xi_s)\,ds\right]\psi(\xi_s)f(\xi_s)\,ds\right) \quad (2.18)$$

where W_x denotes Wiener measure on the coordinates $\xi_\tau(w)=w(\tau)$ of the solution of the stochastic differential equation

$$d\xi_\tau = dW_\tau + d\ln\psi(\xi_\tau)\,d\tau \tag{2.19}$$

We define $P_x = Z_\tau \cdot W_x|\mathscr{F}_\tau$, where $\mathscr{F}_\tau$ denotes the smallest σ-algebra for which the coordinate functions $w(\tau)$ are measurable, and Z_τ, for each $\tau > 0$, is defined by

$$Z_\tau = \psi(x)^{-1}\psi(\xi_\tau)\exp\left[-\int_0^\tau V_\psi(\xi_s)\,ds \,|\, \xi_\tau = y\right] \tag{2.20}$$

Recalling that we assumed that $d\ln\psi$ and $\Delta_g\psi/\psi$ belong to $l^2_{\text{loc}}(\Omega, dx)$, then Z_τ is a random variable, positive W_x a.s., and $E_W(Z_\tau) = 1$. It follows that P_x is a probability measure on $(\mathscr{W}(M), \mathscr{F})$. Then, the diffusion process defined by

$$B_\tau = \xi_\tau - \int_0^\tau d\ln\psi(\xi_s)\,ds \tag{2.21}$$

is proved to be standard Brownian motion starting at x with respect to the measure P_x, so that under this measure, the coordinate maps of the process ξ_τ are the unique solutions to the equation obtained upon differentiating (2.21), which is (2.19), and they behave like a Brownian motion plus the drift. This is the Girsanov or *drift* transformation produced by the unique transformation of probability given by $W_x \to P_x$.

Therefore, for all $\tau \geq 0$ and ξ starting at x, we have

$$E_{P_x}(f(\xi_\tau)) = E_{W_x}(f(\xi_\tau)Z_\tau) \tag{2.22}$$

so that the transition density with respect to the Riemannian volume element is

$$p^\psi_\tau(x, y) = \psi(x)^{-1}\psi(y)E_{W_x}\left(\exp\left[-\int_0^\tau V_\psi(\xi_s)\,ds \,|\, \xi_\tau = y\right]\right)p_\tau(x, y) \tag{2.23}$$

where p_τ is the transition probability of the standard Brownian process,

$$p_\tau(x, y) = (2\pi\tau)^{-2}\exp(-|x-y|^2/2\tau) \qquad (x, y \in M) \tag{2.24}$$

Then, $p^\psi_\tau(x, y)$ is the fundamental solution of the parabolic equation

$$\frac{\partial p^\psi_\tau(x, y)}{\partial\tau} = H_\psi(x)p^\psi_\tau(x, y)$$

Under the assumption that M is compact, it can be proved that $p^\psi_\tau(x, y)$ converges when τ goes to infinity, uniformly on x and y to $\psi^2(y)$, with an exponential uniform convergence on x.

We can extend the diffusion processes to $\mathbb{R}$, by defining $\xi^*(\tau)=\xi(-\tau)$. We would obtain a relativistic extension of the *osmotic processes* defined by Nelson (1985).

A final remark. We have built the chain of one-to-one correspondences by taking as the primitive point of view that of a gauge theory of the gravitational field, introduced by the conformal-Lorentz (-orthogonal) group, or its generalization to arbitrary dimension, for which the Weyl torsion 1-form has an (apparent) universality, as we showed in the consequent constructions.

Yet, when considering the field equations for such a gauge theory, and its relation to global invariants of the configuration spaces [which, in fact are very strongly linked with the asymptotic properties of p_τ^Ψ for $\tau \to 0$, the high-temperature limit of quantum statistical mechanics (Hurt, 1983)], the dimension equal to 4 is singled out among all others. This seems to indicate that if there is to be a description of quantum phenomena, taking as primitive a gauge theory of the gravitational field, the dimension of space-time must be 4 (Rapoport-Campodónico, 1991), and consequently, the description of many degrees of freedom will be essentially relativistic.

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